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DIFFUSION MODELS OF OPTION PRICING: REVIEW OF SOME LOCAL VOLATILITY MODELS (LVM) AND STOCHASTIC VOLATILITY MODELS (SVM) WITH COMPUTATIONAL EXAMPLES

Abstract

This paper is about financial models based on diffusion. And such models are represented by the Stochastic differential equation (SDE) driven by a Brownian motion. In this paper we are considering local volatility models and stochastic volatility models. These models are solving the shortcomings of Black-Scholes model namely by assuming that the volatility of the underlying price is a stochastic process rather than a constant, it becomes possible to model derivatives more accurately. Models like Carr-Madan and Black-Scholes Fourier pricing were presented before the previous two classes of models. In the class of stochastic volatility models SABR STOCHASTIC α, β, ρ model along with Heston model and displaced diffusion DD models are their main representatives in this paper. When it comes to market and model comparison this paper concludes that SABR model, Displaced diffusion (DD) model and Heston model are very close to market results, when it comes to implied volatility and strike price (SABR, DD) and Heston model are better when compared implied volatility with moneyness (strike price /spot price).

Keywords: diffusion models, local volatility, stochastic volatility, SABR model, Heston model

JEL: G12, G13

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1. INTRODUCTION

This paper will deal with financial models that are based on diffusion processes. While the Black-Scholes(1973) model is the simplest formulation for derivative pricing and is still utilized, there is a flaw of that model when volatility surfaces, a situation which implies different underlying parameters for every quoted option, so in that situation Black-Scholes(1973)model is unable to correctly predict the evolution of prices of the underlying asset, see Hirsa (2012). But the first explicit general equilibrium solution to the option pricing problem for simple puts and call was presented in Black,Scholes(BS) (1973) and Merton (BSM) (1973) all this four paper by Merton 1973 a,Merton 1973 b, Merton1973 c ,Merton 1975 , provides, within the Capital Asset Pricing Model(CAPM) framework, an elegant answer to the problem of assigning price to every option by identifying a relation between the value of the stock and its option. Important information is modeled as a jump-process because it arrives at discrete times, Merton (1975). These models in order to be consistent with the Efficient market hypothesis (EMH) see Fama (1970) (i.e. that asset prices fully reflect the information), the unanticipated part of the stock price movements should be a martingale (conditional expectation of the next value of the sequence, given all prior information, is equal to the present value). The relationship between implied volatility and exercise price is not constant and may look like a smile, a skew, etc. (for simplicity are all called "smiles") see Orlando,G., Taglialatela(2017). Implied volatility is calculated by taking the observed option price in the market and a pricing formula such as the Black-Scholes formula that will be introduced below and backing out the volatility that is consistent with the option price given other input parameters such as the strike price of the option, for example ,see Kosowski,Neftci (2015). [1]

[1] There is a distinction between implied volatility and actual volatility, later is realized volatility.

Local volatility models can be traced back to the work by Dupire, B. (1994) and Derman, E. ,Kani, I. (1994). The Black-Scholes theory relies on two assumptions: the values of contingent claims do not depend on investor preferences; therefore, the option can be valued as though the underlying stock's expected return is riskless. [2]Second assumption, is that stock prices evolve log-normally with constant local volatility σ_{ν}. [3] But market option prices are not consistent with Black-Scholes formula. Some models and formulas that account better for volatility when stock price is close to strike price include: Brenner, Subrahmanyam (1998), Bharadia, Christofides and Salkin Formula (1995). For these previous two formulas the accuracy of the approximation worsens as soon as the option departs from the at-the-money position (ATM). Next, this paper will introduce a class (not all of them) of stochastic volatility models that are extending the classic Black-Scholes or local volatility framework. These models are modeling not just the skew but the smile also. Overall, the more out-of-the-money(OTM) (Spot- Strike < 0) a call (put) option is, the higher is the corresponding implied volatility. This well-established empirical fact is known as the volatility smile, or volatility skew, and has major implications for hedging, pricing, and marking-to-market of many important instruments. In statistics, stochastic volatility models are those in which the variance of a stochastic process is itself randomly distributed. Here we will review and set computational examples for: SABR model, Heston model, Displaced diffusion model and implied volatility by Newton-Rapshon method. But paper first starts with Black-Scholes Fourier pricing and Carr-Madan (1999) method as representatives of option pricing by transform techniques and direct integration.

[2] The risk neutral valuation is allowed because the option can be hedged with stock to create instantaneously riskless portfolio.

[3] The stock evolution is described simple as: $\frac{dS}{S} = \mu dt + \sigma dW$, where μ is the expected return $\mu = r - q$,risk free rate minus dividend, S is the stock price, dW is a Wiener process $W \sim (0, dt)$

2. BLACK-SCHOLES FOURIER PRICING

2.1 Characteristic function, Levy process, Fourier transform

Characteristic function of payoff is available analytically for all levy processes. [4] **Levy process**- L let be is an infinite divisible random variable $\forall t \in [0, \infty]$

- ✓ L can be written as the sum of a diffusion, a continuous Martingale, [5] and a pure jump process; i.e:

$$L_t = at + \sigma B_t + \int_{|x|<1} xd\widetilde{N}_\tau + \int_{|x|\geq 1} xdN_\tau\,(\cdot, dx), \forall t \geq 0 \qquad (1)$$

In previous expression $\in \Re$, B_t is the standard Brownian motion, N is defined to be the Poisson random measure of the Lèvy process.

- ✓ Lèvy -Khintchine formula: from the previous property it can be shown that for $\forall \tau \geq 0$ one has that :

$$E\left|e^{inL_t}\right| = e\hat{}(-\tau\psi(u)$$

$$\psi(u) = -iau + \frac{\sigma^2}{2}u^2 + \int_{|x|\geq 1}\left(1 - e^{iux}\right)dv(x) + \int_{|x|<1}\left(1 + e^{iux} + iux\right)dv(x)$$

(2)

$a \in \Re; \sigma \in [0, \infty); v > 0$ borel measure and σ is Lèvy measure. More so $v(\cdot) = E[N_1(\cdot, A)]$See Applebaum (2004).Log normal process that we are considering here is:

$$\log\frac{S_{t+\Delta t}}{S_t} \sim \mathcal{N}\left(\left(r - q - \frac{1}{2}\sigma^2\right)\Delta t, \sigma^2\Delta t\right)$$

(3)

Where $S_t$is stock price at time t and σ is stock price volatility,r is a risk-free interest rate and q is the dividend rate. Characteristic function is given as:

$$\psi(\xi) = \exp\left[i\xi\left(r - q - \frac{1}{2}\sigma^2\right)\Delta t - \frac{1}{2}\xi^2\sigma^2\Delta t\right] \qquad (4)$$

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[4] Characteristics function of any real-valued random variable completely defines its probability distribution. Sometimes characteristic functions are denoted by using so called Iverson bracket see Iverson (1962) or as an indicator function $F_x(x) = E\left(\mathbf{1}_{\{X\leq x\}}\right)$,see Abramowitz, M. ,Stegun (1972).

[5] A sequence of random numbers $X_0, X_1, ..$ with finite means and conditional expectation of $X_{n+1}|X_0, \dots, X_n = X_n$ i.e., $\langle X_{n+1}|X_0, \dots, X_n\rangle = X_n$

Or in general characteristic function for random variable X is given as:
$\varphi: \mathbb{R} \to \mathbb{C}$, and :

$$\varphi(u) \mapsto \mathbb{E}[\exp(iuX)] = \int_{-\infty}^{+\infty} \exp(iux) dF(x) \tag{5}$$

The characteristic exponent is logarithm of the characteristic function,[6] or $\psi = \log\big(\varphi(u)\big)$,and the n-th moment of the random variable if $\exists \mathbb{E}[X^n]$ is :

$$\mathbb{E}[X^n] = i^{-n} \frac{d}{du^n} \varphi(u)|_{u=0} \tag{6}$$

See Kienitz, Wetterau (2012) on this part, here also cumulant function k, moment generating function, θ, and cumulant characteristic function are given as:

$$\begin{cases} k(u) = \log\big(\varphi(iu)\big) \\ \theta(u) = \varphi(iu) \\ \varphi(u) = \log \varphi(u) \end{cases} \tag{7}$$

Where $i = \sqrt{-1}$ is an imaginary number, the payoff for European vanilla option,g is given as:

$$g(S_T) = \max(\theta(S_T - K), 0) \; = \big(\theta(S_T - K)\big)^+ \tag{8}$$

Where S_T is the stock price at maturity T, K is the strike price and $\theta = 1$ for call and $\theta = -1$ for put. Payoff is equal to log of price: $x = \log \frac{S_T}{S_0}$; log strike is:$k = \log \frac{K}{S_0}$,lower log-barrier is: $l = \log \frac{L}{S_0}$; upper log-barrier is : $u = \log \frac{U}{S_0}$.
So now the payoff function becomes:

$$g(x) = e^{ax} S_0 \Big(\theta(e^x - e^k)\Big)^+ \mathbb{1}_{[lu]}(x) \tag{9}$$

[6] CDF: $F(x) = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{iux}\varphi(-u) - e^{-iux}\varphi(u)}{iu} du$; PDF:$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \varphi(u) du = \frac{1}{\pi} \int_0^{\infty} e^{iux} \varphi(u) du$

The Fourier transform is given as:

$$\hat{g}(\xi) = \mathcal{F}_{x\to\xi}[g(x)] = \int_{\mathbb{R}} e^{i\xi x} e^{ax} S_0\left(\theta(e^x - e^k)\right)^+ \mathbb{1}_{[lu]}(x)\, dx = S_0 \int_l^u e^{(i\xi+a)x}\left(\theta(e^x - e^k)\right)^{+dx} = S_0 \int_v^\eta e^{(1+i\xi+a)x} dx + S_0 \int_v^\eta e^{k+)i\xi+a)x} dx = S_0 \left(\frac{e^{(1+i\xi+a)\eta} - e^{(1+i\xi+a)v}}{1+i\xi+a} - \frac{e^{k+(i\xi+a)\eta} - e^{k+(i\xi+a)v}}{i\xi+a}\right) \quad (10)$$

Where $\eta = \begin{cases} u\ for\ call \\ l\ for\ put \end{cases}$ and $v = \begin{cases} \max(k,l) for\ call \\ \min(k,u) for\ put. \end{cases}$.In order to find the option value V we need to discount the expected payoff :

$$V = e^{rT}\mathbb{E}\left[g(X_T)e_T^{-aX_T} | X_0 = 0\right] = e^{-rT}\int_{\mathbb{R}} g(X_T)e_T^{-aX_T} f_x(x,T)dx \quad (11)$$

Where $e_T^{-aX_T}$ I the undamping factor and f is the PDF of X_t .Now thanks to Parseval-Plancherel theorem integral can be computed in Fourier space:

$$V = \frac{e^{-rT}}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)\psi^*(\xi + ia, T)d\xi \quad (12)$$

Where $\psi(\xi, t) = \hat{f}_x(x,t)$ is the characteristic function of X_t , and * denotes complex conjugate, we can include undamping $\psi^*(\xi + ia, T)$factor due to shift theorem

2.2 Parseval-Plancherel theorem

Plancherel theorem (sometimes called the Parseval–Plancherel identity) is a result due to Plancherel (1910).

Theorem 1 : Parseval–Plancherel theorem

$$\int_{\mathbb{R}} f(x)g^*(x)dx = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}^*(\xi)d\xi \quad (13)$$

Proof*: in the LHS we have :*

$$f(x) = \mathcal{F}^{-1}_{\xi\to x}[\hat{f}(\xi)] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}\hat{g}^*(\xi)d\xi$$
$$g(x) = \mathcal{F}^{-1}_{\xi\to x}[\hat{g}(\xi)] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}\hat{g}(\xi)d\xi \quad (14)$$

Since $(ab)^* = a^*b^*, \left(e^{-ix\xi}\right)^* = e^{+ix\xi}$ and so we have: $g^*(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{g}^*(\xi)d\xi$and henceforth :

$$\int_{\mathbb{R}} f(x)g^*(x)dx = \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-ix\xi}\hat{f}(\xi)d\xi\right)\left(\int_{\mathbb{R}} e^{ix\xi'}\hat{g}^*(\xi')d\xi'\right)dx = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\hat{f}(\xi)\hat{g}^*(\xi')\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i(\xi-\xi')x}dxd\xi'd\xi \quad (15)$$

Since $\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i(\xi-\xi')x}dx = \delta(\xi-\xi')$, where δ is Dirac delta see Dirac (1958).[7]

$$\begin{cases} \int_{\mathbb{R}} f(x)g^*(x)dx = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(\xi)\int_{\mathbb{R}}\hat{g}^*(\xi')\delta(\xi-\xi')d\xi'd\xi \\ \int_{\mathbb{R}}\hat{g}^*(\xi')\delta(\xi-\xi')d\xi' = \hat{g}^*(\xi) \\ \int_{\mathbb{R}} f(x)g^*(x)dx = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(\xi)\hat{g}^*(\xi)d\xi \ \blacksquare \end{cases} \quad (16)$$

2.3 Shift theorem

Theorem 2 : Shift theorem

$$\mathcal{F}_{x\to\xi}[f(x)e^{-ax}] = \hat{f}(\xi+ia) \quad (17)$$

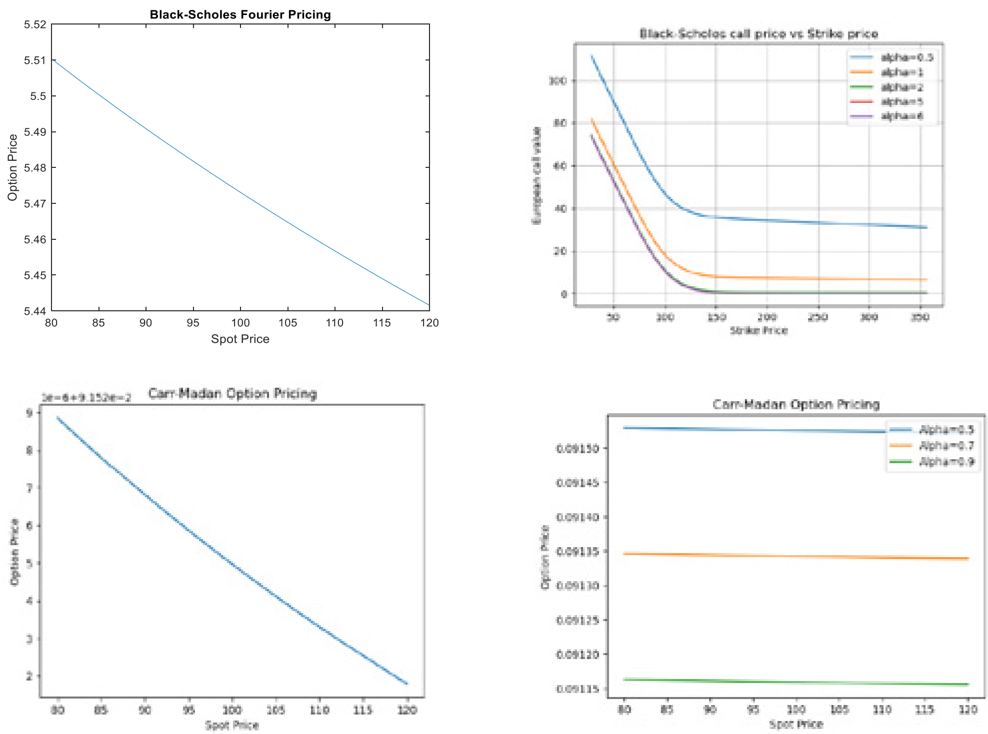
Proof:

$$\mathcal{F}_{x\to\xi}[f(x)e^{-ax}] = \int_{\mathbb{R}} e^{ix\xi}e^{-ax}f(x)dx = \int_{\mathbb{R}} e^{i(\xi+ia)x}f(x)dx = \hat{f}(\xi+ia)\blacksquare \quad (18)$$

Where we were using the fact that $i^2 = -1$.

[7] The delta function is sometimes called Dirac's delta function or the "impulse symbol" ,Bracewell (2000),delta function can be viewed as derivative of Heaviside step function: $\frac{d}{dx}[H(x)] = \delta(x)$,and has a fundamental property $\int_{a-\epsilon}^{a+\epsilon} f(x)\delta(x-a)dxf(a), \forall\epsilon = 0$

Figure 1. Black -Scholes method Fourier pricing and Carr-Madan option pricing with one and more α dampening parameters



Source: Authors own calulation inPython and code available at: https://github.com/MalutiKgarose/Option-prices-using-FFT/blob/master/FFTvsAnalytical%20Black-Scholes.ipynb

Carr-Madan (1999) model, call price is given as: $C(K,T) = \int_k^{\infty} e^{-rT}(e^y - e^k) f(y|x)dy$; $C(K,T) \rightarrow S(0); K \rightarrow 0; \vee\, k \rightarrow \infty$ Parameter α is called dampening parameter.

And the call price will be : $c(k,T) \coloneqq e^{\alpha k} C(K,T)$. Fourier transform of call function is: [8]

$$\varphi_{call}(v) = \int_{-\infty}^{+\infty} e^{ivk} c(k,T) dk \tag{19}$$

[8] Upper bound is: $\alpha^{VG}\_\sup = -\frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 v}} - 1$ where VG is variance gamma model, and $\alpha_{sup}^{VGGOU} = -\frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 v}(1 - \exp\left(\frac{v\beta\lambda}{1-e^{-vt}}\right)} - 1$

In the case of European call option :

$$\varphi_{call}(\nu) = \int_{-\infty}^{+\infty} e^{i\nu k} \int_{k}^{+\infty} e^{-rT+\alpha k}(e^{y} - e^{k}) f(y|x) dy dk$$
$$= \frac{e^{-rT}\varphi(\nu - (\alpha+1)i)}{\alpha^2 + \alpha - \nu^2 + i(2\alpha+1)\nu}$$

When $k = \log(K)$:

$$C(k,T) = \frac{e^{-\alpha k}}{\pi}\int_0^{\infty} e^{-i\nu k}\,\varphi_{call}(\nu) d\nu \tag{20}$$

Black-Scholes analytical formulas will help us to plot put and call surfaces here. The value of call option at time t is equal to:

$$C(S,t) = S_t e^{-qt}\, N(d_1) - Ke^{rt}N(d_2) \tag{21}$$

Where :

$$\begin{cases} d_1 = \frac{\log\frac{S_t}{K} + \left(r - q + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \\ d_2 = d_1 = -\sigma\sqrt{\tau} \end{cases} \tag{22}$$

Where S_t is the stock price at time , T is the expiration rate, τ is time to maturity, i.e. $\tau = T - t$, K is the strike price,r is the risk free interest rate . q is the dividend rate, σ is the stock volatility $\mathcal{N}$ is the CDF of cumulative standard normal distribution function defined as:

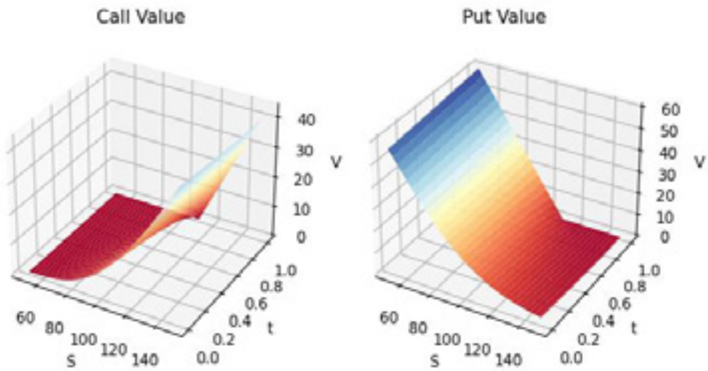
$$\mathcal{N}(x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-\frac{1}{2}\phi^2} d\phi \tag{23}$$

The value of a put option is given as:

$$P(S_t,t) = Ke^{-rt}\mathcal{N}(-d_2) - S_t e^{-qt}\mathcal{N}(-d_1) \tag{24}$$

Next we are plotting Call-Value and Put-value and (S,T,v)

Figure 2. Call value , Put value and S, t, v



Source: Author's calculations based on a code available at: https://github.com/Robin-Guilliou/Option-Pricing/tree/main/European%20Options

3. SABR MODEL (STOCHASTIC α, β, ρ MODEL) AND DISPLACED DIFFUSION (DD) MODELS

In this part we are taking into consideration SABR or stochastic volatility (stochastic α, β , ρ) model introduced in Hagan, P.S., Kumar, D., Lesniewski, A.S., Woodward, D.E. (2002). SDEs of the model are given as:

$$\begin{cases} dS_t = \sigma_t S_t^{\beta} dW_t \\ d\sigma_t = \sigma_t v dZ_t \\ S(0) = S_0 \\ \sigma(0) = \sigma_0 \\ \langle DW_t, dZ_t \rangle - \rho dt \end{cases} \tag{25}$$

Here S_0 is the spot asset price and σ_0 is the spot value of volatility. The other model parameters are the CEV parameter (constant elasticity of variance) β,[9] the volatility of volatility ν and the correlation ρ between the Brownian motions W and Z driving the asset and the volatility dynamics. The original SABR pricing formulae is given as:

$$\sigma_{SABR}(K,T) \approx \frac{\sigma_0}{(SK)^{\frac{1-\beta}{2}}\left(1+\frac{(1-\beta)^2}{24}\log^2\left(\frac{S}{K}\right)+\frac{(1-\beta)^4}{1920}\log^4\left(\frac{F}{K}\right)+..\right)}\frac{z}{x(z)}\left(1+\left(\frac{(1-\beta)^2\sigma_0^2}{24(SK)^{1-\beta}}+\frac{\rho\beta\nu\sigma_0}{4(SK)^{\frac{1-\beta}{2}}}+\nu^2\frac{2-3\rho^2}{24}\right)T+\cdots\right) \quad (26)$$

Where $= \frac{\frac{\nu}{\alpha}(fK)^{\frac{1-\beta}{2}}logf}{K}$; and $x(z) = \log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}$ for the special case of ATM (at the money) options :

$$\sigma_{ATM} = \sigma_B(f,f) = \frac{\alpha}{f(1-\beta)}\left\{1+\left[\frac{(1-\beta)^2}{24}+\frac{\alpha^2}{f^{2-2\beta}}+\frac{1}{4}\frac{\rho\beta\alpha\nu}{f(1-\beta)}+\frac{2-3\sigma^2}{24}\nu^2\right]t_{ex}+.\right\} \quad (27)$$

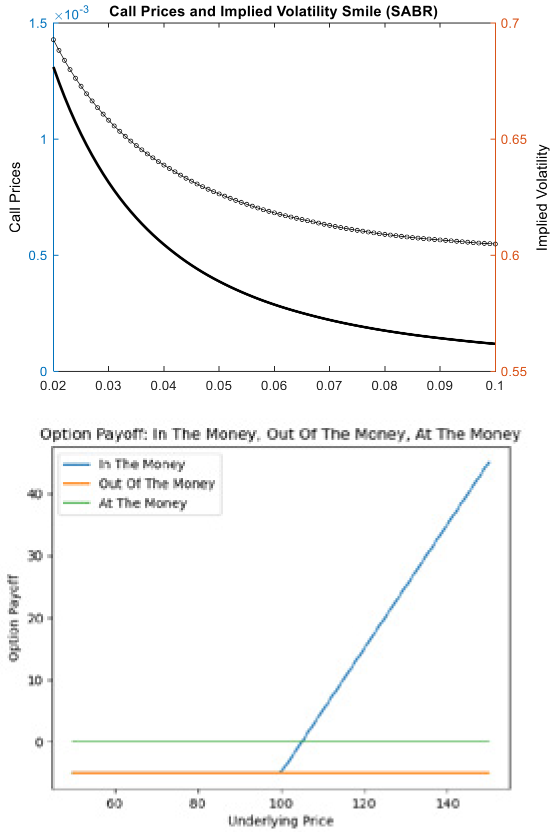
See Hagan, P.S., Kumar, D., Lesniewski, A.S., Woodward, D.E. (2002) . Next we will graphically depict SABR model with implied volatility :$d\,S_t = r\,S_t dt + \boldsymbol{\sigma}\,S(t)dW\,(t)$,and
σ is implied volatility, and $\sigma \approx I^0(x)(1+I^1(x)\tau)+O(\tau^2)$; where:

$$I^0(x) = \begin{cases} \alpha K\beta^{-1}\,;x=0 \\ \frac{xa(1-\beta)}{S^{1-\beta}-K^{1-\beta}} \\ \nu x\log\left(\frac{\sqrt{1-\rho z_1+z_1^2}+z_1-\rho}{1-\rho}\right);\beta=1\,; \\ x\log\left(\frac{\sqrt{1-\rho z_1+z_1^2}+z-\rho}{1-\rho}\right);\beta<1 \end{cases} \quad (28)$$

[9] Constant elasticity of variance (CEV)model is a stochastic volatility model that attempts to capture stochastic volatility and the leverage effect. The standard CEV model : $dS_t = \mu S_t dt + \sigma_{cev}\,S_t^{\beta}dW_t$, $S(0) = S_0$.This model is due: Schroder, M. (1989) and Andersen, L. ,Andreasen, J.(2000), see Kienitz, Wetterau (2012)

And $I^1(x) = \frac{(\beta-1)^2}{24}\frac{\alpha^2}{(SK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\alpha\nu}{(SK)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24}\nu^2$ and where $z_1 = \frac{\nu x}{\alpha}$; $z = \frac{\nu}{\alpha}\frac{S^{1-\beta}-K^{1-\beta}}{1-\beta}$

Figure 3. Call prices and implied volatility SABR and OTM, ATM,ITM



Source: Author's calculations and code provided by : https://de.mathworks.com/matlabcentral/profile/authors/3467507
DD model had been presented by Rubisnstein (1983).

DD model can be presented in following manner:

$$DS_t = \mu(S_t + a)dt + \sigma_{DD_t}(S_t + a)dW_t; S(0) = S_0 \qquad (29)$$

The only parameter different than the standard Black-Scholes(1973) model is $a > 0$.This is called displacement parameter hence the name of the pricer.Pricng formulae in DD model is given as see also Rebonato (2002):

$$C(K,T) = e^{-rT}\left((S(0)+a)\mathcal{N}(d_1) - K^*\mathcal{N}(d_2)\right)$$
$$P(K,T) = e^{-rT}\left(K^*\mathcal{N}(-d_2) - (S(0)+a)\mathcal{N}(-d_1)\right) \quad (30)$$

Where $K^* = K + a$ and where :

$$d_1 = \frac{\log\left(\frac{S(0)+a}{K^*}\right)+\frac{\sigma_{DD}^2}{2}T}{\sigma_{DD}\sqrt{T}}; d_2 = d_1 - \sigma_{DD}^2 T \quad (31)$$

For time dependent volatility we replace σ_{DD}^2 with $\nu_{DD}^2(t_0,t_1) \coloneqq \int_{t_0}^{t_1}\sigma_{DD}^2 u(du)$,parity between Black-Scholes and DD model means : $C_{DD}(K,T) = C_{BS}(K,T)$.Rebonato (2004) shows that European call option ATM (at the money) prices can be recovered reasonably:

$$\sigma_{DD} \approx \frac{S_0}{S_0+a}\sigma_{BS}\frac{1-\frac{1}{24}\sigma_{BS}^2 T}{1-\frac{1}{24}\left(\frac{S_0}{S_0+a}\sigma_{BS}\right)^2 T} \quad (32)$$

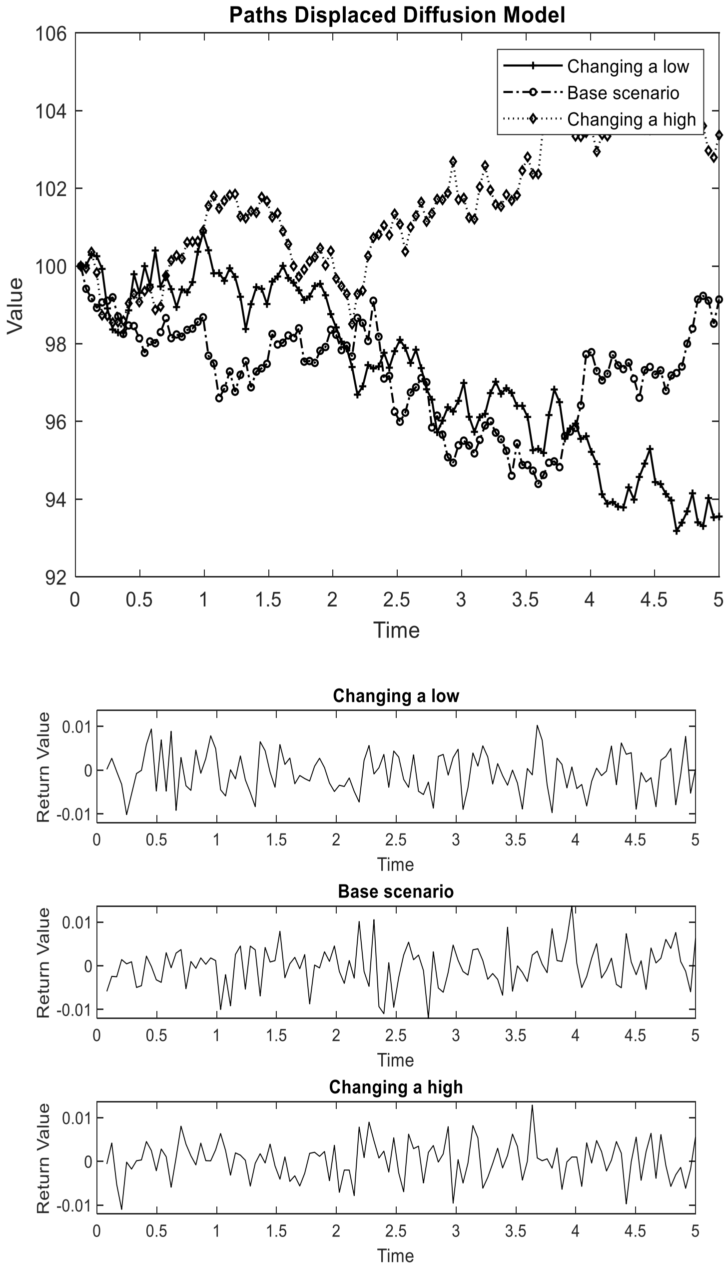
Here as in Marris(1999) and Rubinstein (1983), Geske (1977),arbitrage pricing mechanism leads to European option call formula: $E[ae_t^W S_0 + bS_0 - K]^+$; [10] which resembles Cox,Ross (1976): $E[e^z S_0 - K]^+$ and now the analytical solution as an adjustment to Black-Scholes formula is:

$$PV_{eurocall}(S_0,K,t,r,\sigma) \rightarrow PV_{eurocall}(aS_0,K-bS_0,t,r,\sigma_r) \quad (33)$$

Rubinstein (1983) requires debt to be riskless which means $\alpha < 1$,if so firms debt will not exceed firms riskless assets, which is opposite from $\alpha > 1$ where firm's debt exceed its riskless assets see Hull (1997).Next we are plotting DD model and DD SABR model.

[10] Here also:$a = \alpha(1+\beta); b = (1-\alpha-\alpha\beta)r$, where α are risky assets, $\beta = \frac{debt}{equity}$

Figure 4. DD model for different $\alpha \in (0.0, 50, 100)$



Source : Author's calculations and code provided by :
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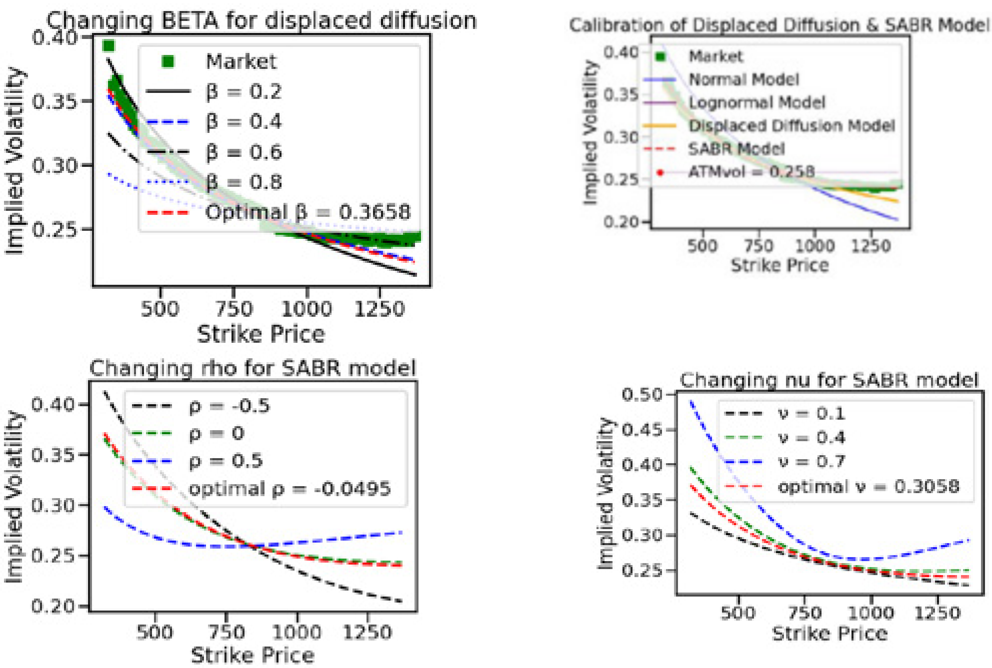
Figure 5. DD SABR model, with changing $\beta; \rho; \nu$

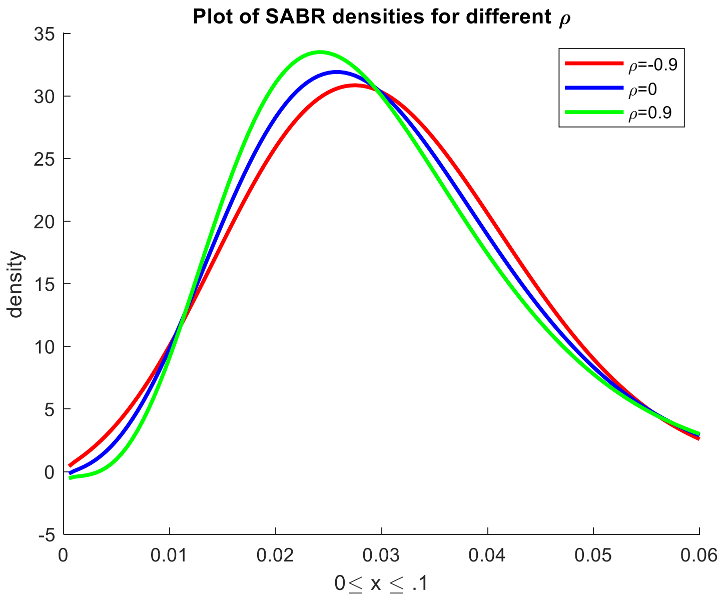


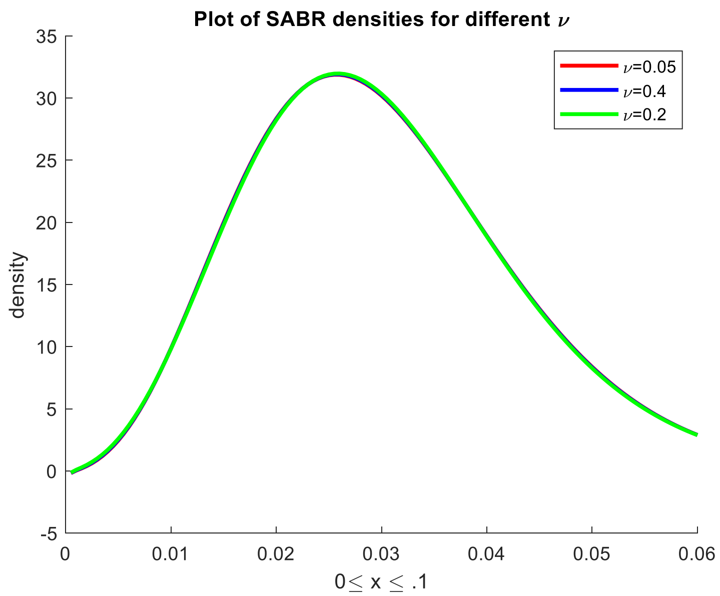
https://github.com/Mordant-Black/options_SABR_model/blob/master/Volatility%20Smile.ipynb

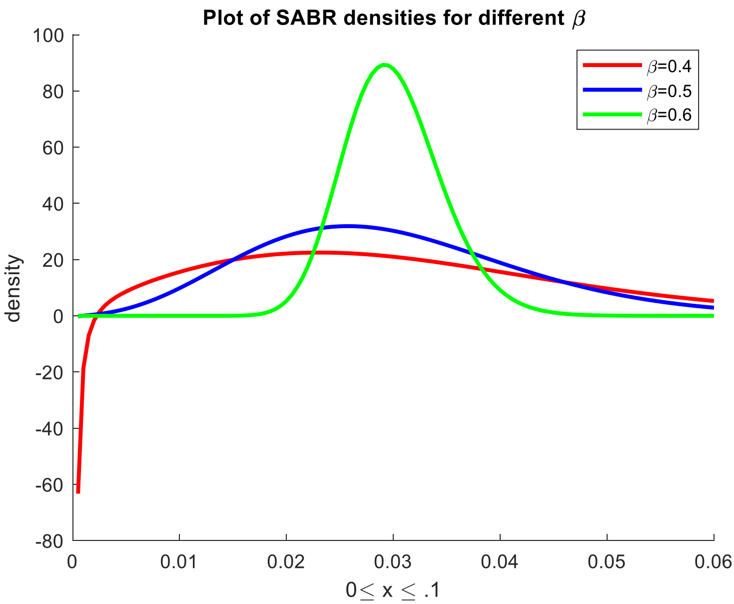
Model Parameters
1) σ: 0.25827414878228955, β: 0.3657704004978997 2) α: 7.468394914810275 , ρ: -0.049519209180403284 , ν: 0.30576503781656666

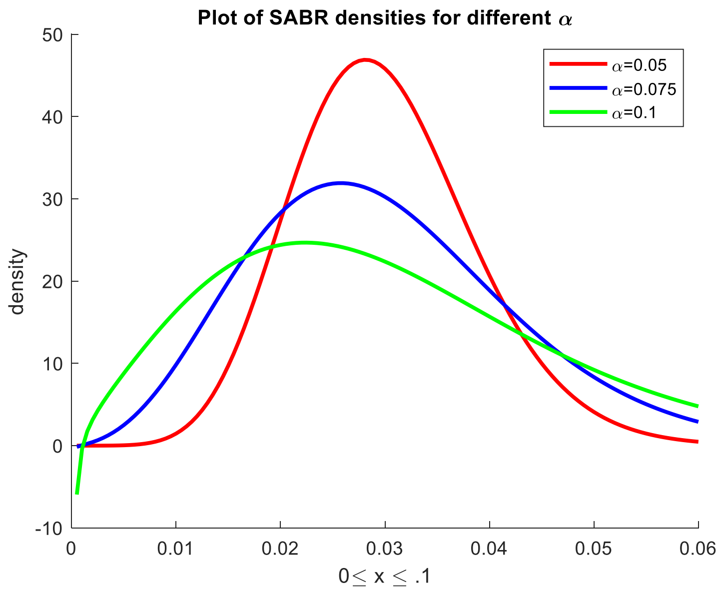
If $\beta = 0$ model is stochastic normal, when $\beta = 1$ model is stochastic log-normal. Next we are plotting CDF's for SABR for different ρ, ν, β, α

Figure 6. CDF's SABR for different ρ, ν, β, α









Source : Author's own calculation based on a code provided at: https://de.mathworks.com/matlabcentral/profile/authors/3467507

4. IMPLIED VOLATILITY: NEWTON-RAPHSON METHOD

In the Newton's method the algorithm can be applied iteratively to obtain:$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_{n-1})}$,if $\lim_{x_{n+1} \to x^*} \frac{f(x_n)}{f'(x_n)} = x_n$,and $x_n = x^* + \epsilon_n$,where $\epsilon_{n+1} = \frac{f''(x^*)}{2 \cdot f'(x^*)} \epsilon_n^2$ .Fixed point theorem states that if $\exists f(x) \in [a, b]$,then $\exists x \in [a, b]$, and $f(x) - x = 0 \Rightarrow f(x) = x$, see Rosenlicht (1968).In our case let V_m denotes the market price of an option,$V_{BS}(\sigma)$ is a price of an option obtained by Black-Scholes model we should have a goal to find volatility σ_I such that :$V_m = V_{BS}(\sigma_I)$ .Now we will use Newton-Raphson technique the initial guess of implied volatility is σ_I^0 afterwards with each step iteration we will improve the result: $\sigma_I^{n+1} = \sigma_I^n + \frac{V_m - V_b(\sigma_I^n)}{\nu(\sigma_I^n)}$ and:

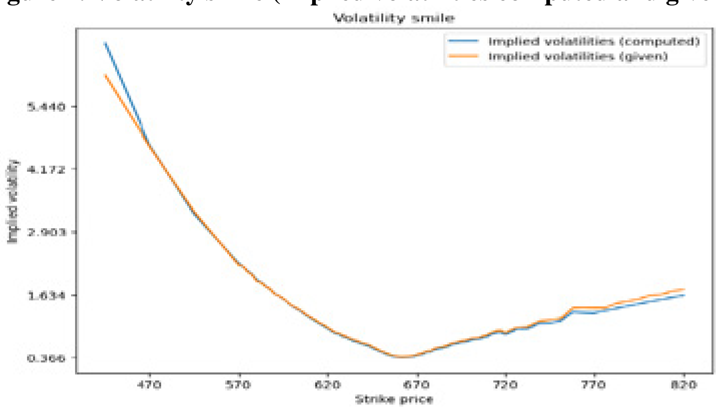
$$\nu = \frac{\partial \mathrm{V_b}}{\partial \sigma} = S_t \sqrt{T - t e}^{-q(T-t)} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{\log \frac{S_t}{K} + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \right)^2 \right] \quad (34)$$

ν is the vega of the option.S_t is the stock price at time t, K is the strike price r is the risk-free rate and q is the dividend rate. Or formally the procedure is defined as:

$$\sigma_{n+1} = \sigma_n - \frac{\mathcal{N}(d_1)S - \mathcal{N}(d_2)Ke^{-rT} - C^*(S,T)}{S\mathcal{N}'(d_1)\sqrt{T}} \ s.t.$$
$$d_1 = \frac{1}{\sigma\sqrt{T}} \left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T \right]; d_2 = d_1 - \sigma\sqrt{T} \quad (35)$$

Next we are plotting volatility smile (implied volatilities computed and given)

Figure 7. Volatility smile (implied volatilities computed and given)



Source: Author's own calculation based on a code provided at: https://github.com/woutervanheeswijk/Implied_volatility_calculator

5. HESTON MODEL

Heston model is due to Heston (1993).In this model spot asset at time t follows diffusion:

$$dS(t) = \mu s dt + \sqrt{v(t)} S dW_1(t)$$
$$dv(t) = k(\Theta - v(t))dt + \sigma_v \sqrt{v(t)} dW(t)^2 \quad (36)$$

In previous $\mu = r - q$; W_1 is a Wiener process,the square of volatility follows a CIR process Cox.Ingersoll.Ross model (1985). [11] In previous following symbols have this meaning:

r is the continuous risk-free rate,q is the continuous dividend yield,$S(t)$ is the asset price at time t, vt is the asset price variance at time t , v_0 is the initial variance of the asset price at $t = 0$ for ($v_0 > 0$),θ is the long-term variance level for ($\theta > 0$). κ is the mean reversion speed for the variance for ($\kappa > 0$),σ_v is the volatility of the variance for ($\sigma_v > 0$),p is the correlation between the Weiner processes W_t and W_t^v for ($-1 \leq p \leq 1$).

[11] CIR model follows process where the short-rate is assumed to satisfy the following differential equation:$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{rt}dw(t)$, where $k, \sigma, \theta > 0$ with $2k\theta > \sigma^2$ and w is an Brownian motion under risk-free measure. In the CIR model the price of a zero-coupon bond with maturity T at the time $t \in [0, T]$ is given as :$P(t,T) = A(t,T)e^{-r(t)B(t,T)}$,where $A(t,T) = \left(\frac{2he^{\frac{(h+k)(T-t)}{2}}}{2h+(h+k)(e^{h(T-t)}-1)}\right)^{\frac{2k\theta}{\sigma^2}}$; $B(t,T) = \frac{2(e^{h(T-t)}-1)}{2h+(h+k)(e^{h(T-t)}-1)}$; $h = \sqrt{k^2 + 2\sigma^2}$

Table 1 Characteristic function Heston model

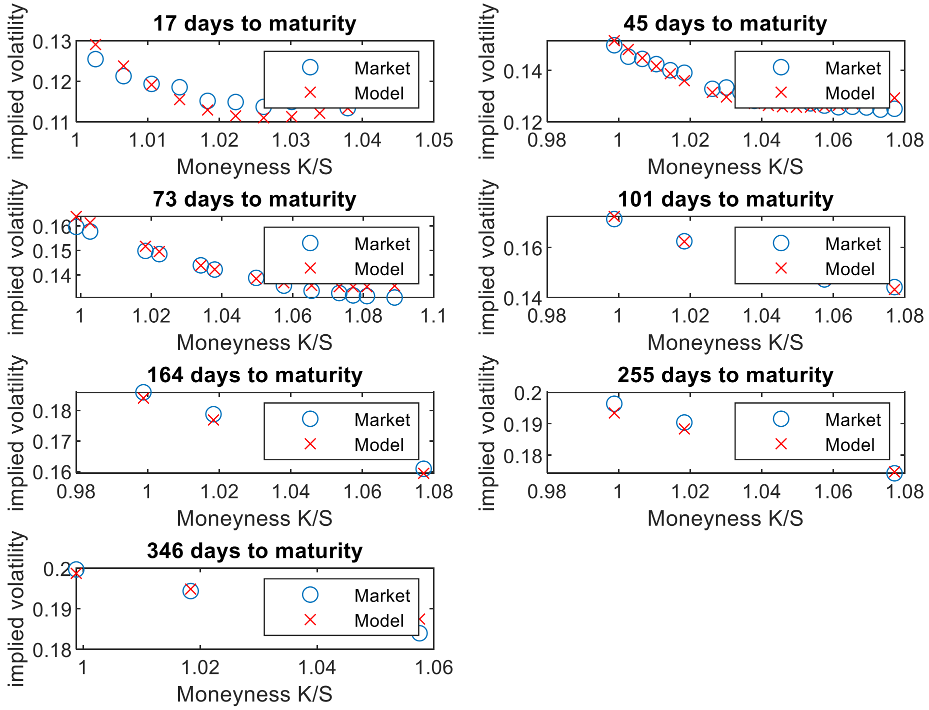
Characteristic function Heston	$f_{Heston_j}(\phi) = \exp\left(C_j + D_j v_0\right) + i\phi \ln S(t))$
Elements of characteristic function (1)	$C_j = (r-q)i\phi\tau + \frac{k\theta}{\sigma_v^2}\Big[(b_j - p\sigma_v i\phi + d_j)\tau - 2\ln\left(\frac{1-g_j e^\wedge(d_j\tau)}{1-g_j}\right)n\Big]$
Elements of characteristic function (2)	$D_j = \frac{b_j - p\sigma_v i\phi + d_j}{\sigma_v^2}\left(\frac{1-e^{d_j\tau}}{1-g_j e^{d_j\tau}}\right)$
Elements of characteristic function (3)	$g_j = \frac{b_j - p\sigma_v i\phi + d_j}{b_j - p\sigma_v i\phi - d_j}$
Elements of characteristic function (4)	$d_j = \sqrt{(b_j - p\sigma_v i\phi)^2 - \sigma_v^2(2u_j i\phi - \phi^2)}$
Inverted characteristic function CDF	$P_j(x,v,T;\ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln K} f_j(x,v,T;\phi)}{i\phi}\right] d\phi$
Call(K)	$Call(K) = S(t)e^{-q\tau}P_1 - Ke^{-r\tau}P_2$
Put (P)	$Put(K) = S(t)e^{-q\tau}P_1 - Ke^{-r\tau}P_2 - Ke^{-r\tau} - S(t)e^{-q\tau}$

Source : see Heston (1993) , Kienitz, Wetterau (2012) and Albrecher et al .(2012)

Next on the following figure we present Heston model with implied volatility and moneyness.[12]

[12] Moneyness is description of a derivative relating its strike price to the price of its underlying asset

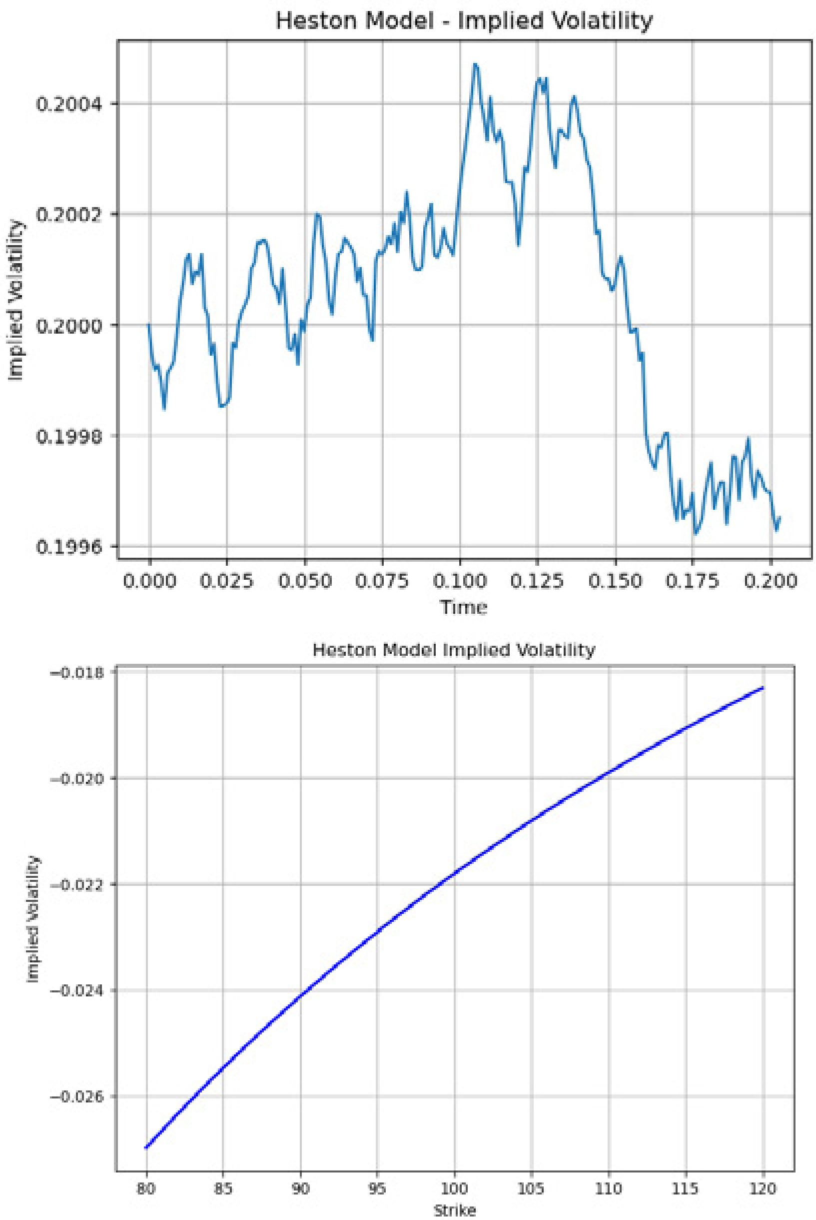
Figure 8. Heston model with moneyness



Source: Authors' calculation based on a code provided at: https://github.com/jcfrei/Heston

next we will graphically depict movement of Heston's model implied volatility and $dt = \frac{T}{N}$ time,

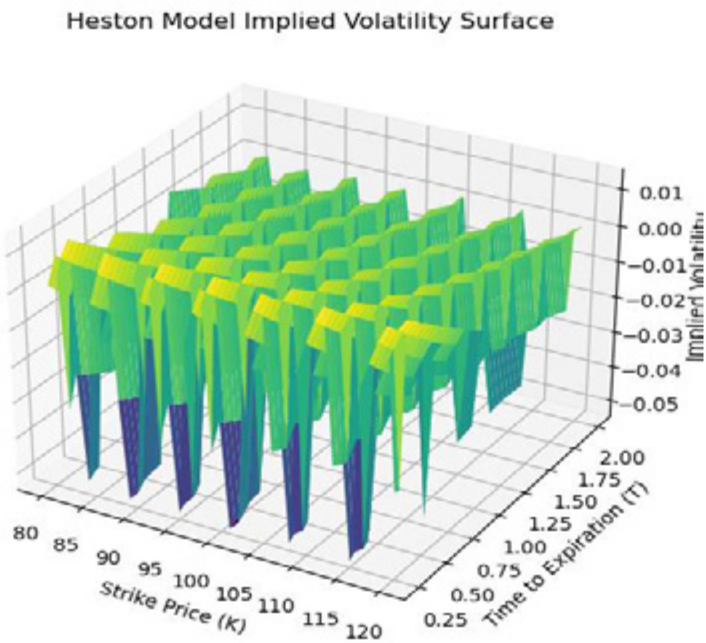
Figure 9. Heston model implied volatility and time and Heston model implied volatility and strike price



Source : Authors own calculation

Parameters on the Heston model implied volatility are: $r = 0.05$ risk free rate ,$\kappa = 2.0$ mean reversion speed, long-term variance $\theta = 0.04$; volatility of volatility $\sigma = 0.3$,$\rho = -0.5$ correlation between asset price and volatility,$\nu_0 = 0.04$ initial volatility, $S(0) = 100$initial asset price ,$N = 1000$ number of time steps, M=1000, Montecarlo paths and parameters for the Heston implied volatility and Strike price are : $\kappa = 2.0$;$\theta = 0.04$; $\sigma = 0.3$; $\rho = -0.8$; $\nu_0 = 0.04$,$r = 0.05$, $S(0) = 100$; $K = 100, T = 1$. Previous two graphs can be plotted in one mesh graph

Figure 10. Heston model implied volatility (IV) surface



Source : Authors own calculation

$\nu_0 = 0.04$,$r = 0.05$, $S(0) = 100$; $K = 100, T = 1$, $\kappa = 2.0$, $\theta = 0.04$, $\sigma = 0.5$, $\rho = -0.5$, $S(0) = 100$; $K = 100, T = 1$

6. Conclusion

This paper confirmed that when it comes to market and model comparison this paper concludes that SABR model, Displaced diffusion (DD) model and Heston model are very close to market results. When it comes to implied volatility and strike price (SABR, DD) and Heston model are better when compared implied volatility with moneyness (strike price /spot price K/S).When β is optimal SABR implied volatility and strike price movements are almost identical to market. In the Heston model for all the levels of moneyness implied volatility is almost identical to actual market volatility or realized volatility. Black-Scholes Fourier pricing and Carr-Madan option pricing proved that there is an inverse relation between striker option prices and spot prices. In the Black-Scholes method the higher α i.e. the risky asset the lower is the value of European call option. A European option is a financial contract that gives the holder a right but not an obligation to buy and sell the underlying asset from the writer at the time of expiry for a pre-determined price. Displaced Diffusion (DD) models are capable of modelling skewed implied volatility structures. In the DD model for different $\alpha \in (0.0,50,100)$ paths generated by the model are continuous since the stochastic driver is a Brownian motion. This model showed that volatility is timely dependent.

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